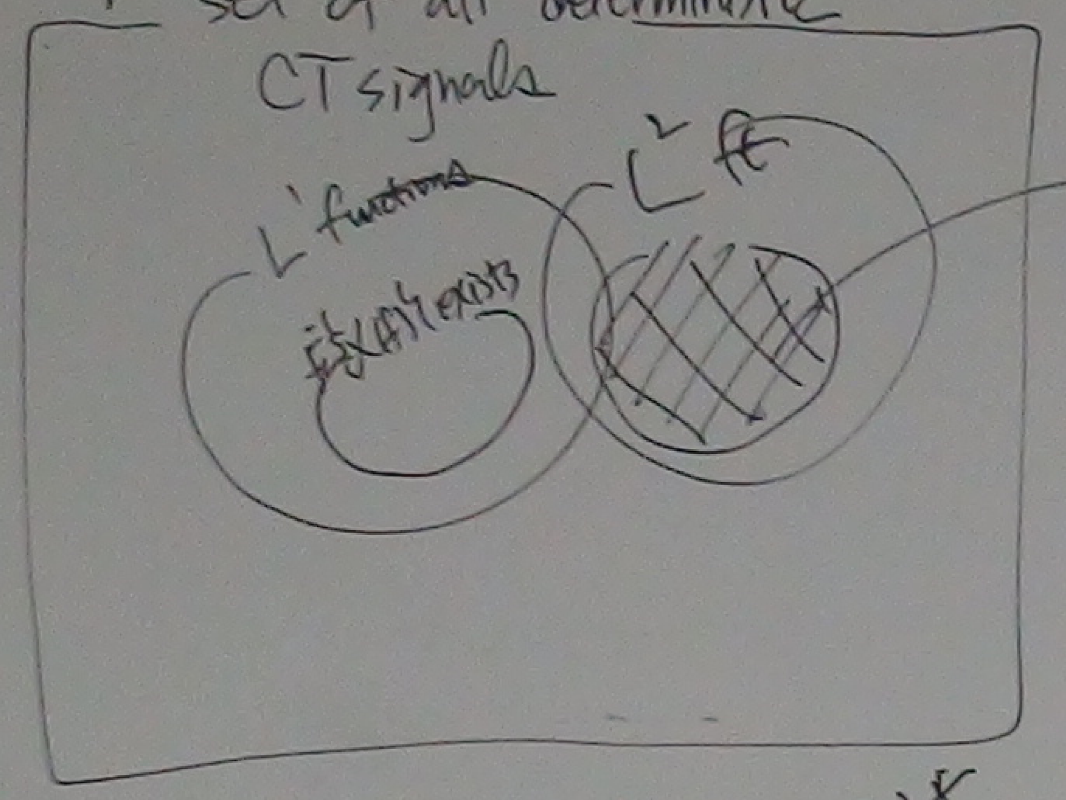
EECE 695M CM#24a

□ FTF

set of all deterministic CT signals



deterministic version of cyclostationary signal

Gardner : ______

Shift-Invariant signals.

(SI)

$$X(f) \triangleq \int_{-\infty}^{\infty} x(t) \left(e^{+j2\pi ft}\right)^* dt$$

subset of L^1 functions

$$L^1 \triangleq \left\{ x(t) : \int_{-\infty}^{\infty} |x(t)|\, dt < \infty \right\} \not\ni \mathrm{sinc}(t)$$

$$L^1 \supset \left\{ x(t) \in L^1 : \int_{-\infty}^{\infty} \boxed{X(f)} e^{j2\pi ft} dt \text{ exists} \right\}$$

$$L^2 \triangleq \left\{ x(t) : \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \right\} \ni \mathrm{sinc}(t) \triangleq \frac{\sin \pi t}{\pi t}$$

Fourier-Plancherel TF.

Theorem. Suppose that $x(t) \in L^2$.

$$x_\Delta(t) \triangleq x(t)\left\{u(t+\tfrac{\Delta}{2}) - u(t-\tfrac{\Delta}{2})\right\}$$

If $x(t)$ bounded,

then $x_\Delta(t) \in L^1, L^2$ &

$$X_\Delta(f) \triangleq \int_{-\infty}^{\infty} x_\Delta(t) e^{-j2\pi ft} dt \text{ exists.}$$

$$\tilde{x}(t) = \lim_{\Delta\to\infty} \left(\int_{-\infty}^{\infty} X_\Delta(f) e^{j2\pi ft} dt \right)$$

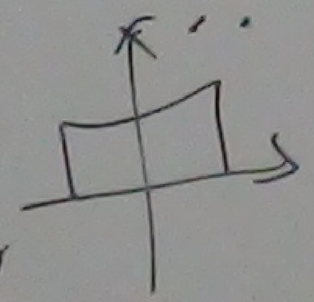
$$X(f) \triangleq \lim_{\Delta\to\infty} X_\Delta(f).$$

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} X_\Delta(f) e^{j2\pi ft} dt - x(t) \right|^2 dt \to 0$$

$\Delta \to \infty$.

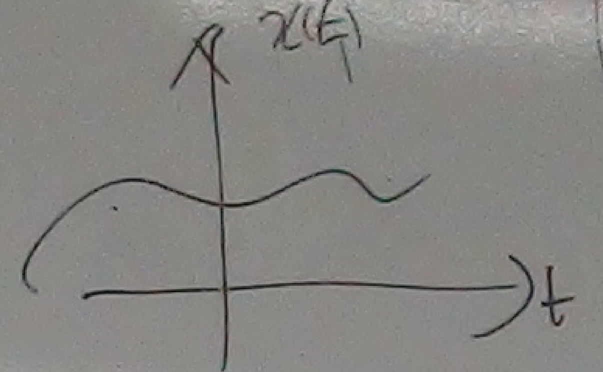
$\therefore$ $x(t) \in L^2 \Rightarrow X(f)$ exists &

$\mathcal{F}^{-1}\{X(f)\} = x(t)$

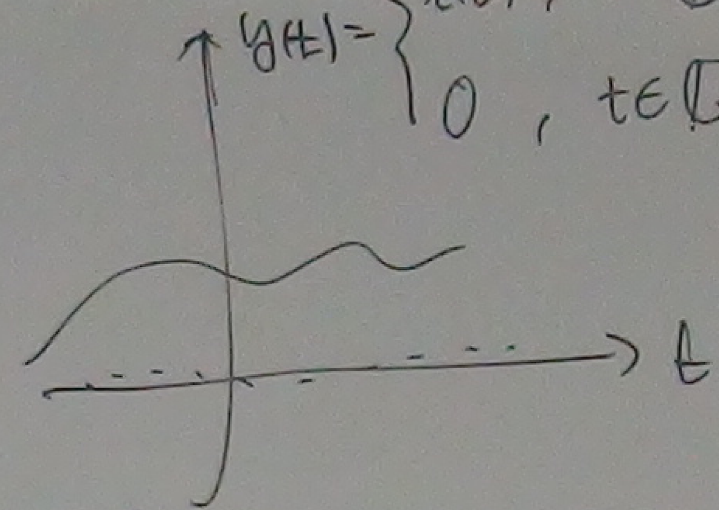


☆ $\displaystyle\int_{-\infty}^{\infty} x(t)\,dt \neq \lim_{\Delta\to\infty} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} x(t)\,dt.$

"excessive rigor"



$$y(t) = \begin{cases} x(t), & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$$



☆ finite-energy signal $x(t)$.

$$S_{XX}(f_1, f_2) \simeq E\{X(f_1) X^*(f_2)\}$$

□ Cyclostationary ~~Signal~~ R.P. $X(t)$

$\simeq$ deterministic periodic signal

$\simeq$ periodic r.p.

(i) $X(t) = X(t+T)$, $\forall t$ a.s.

$$P(\{\omega \in \Omega : X(\omega, t) = X(\omega, t+T),\ \forall t\}) = 1.$$

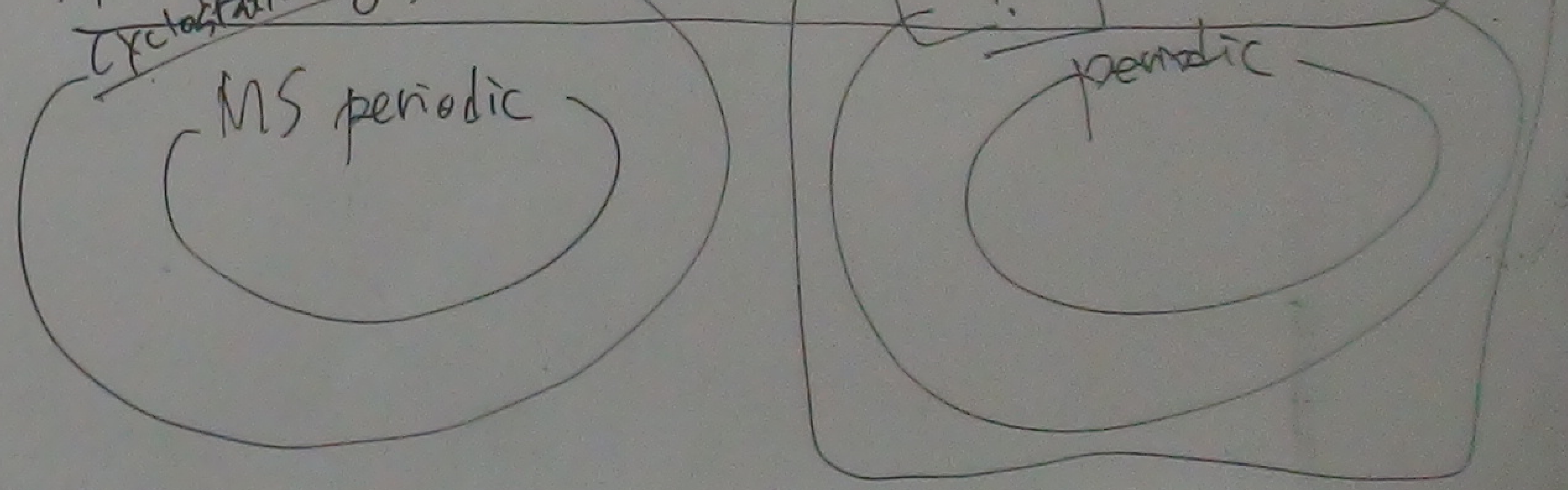
(ii) $X(t) = X(t+T)$, $\forall t$. MS

$$E\{|X(t) - X(t+T)|^2\} = 0,\ \forall t$$

$$\forall t_0 \ (X(t_0) = X(t_0+T) \text{ a.s.})$$

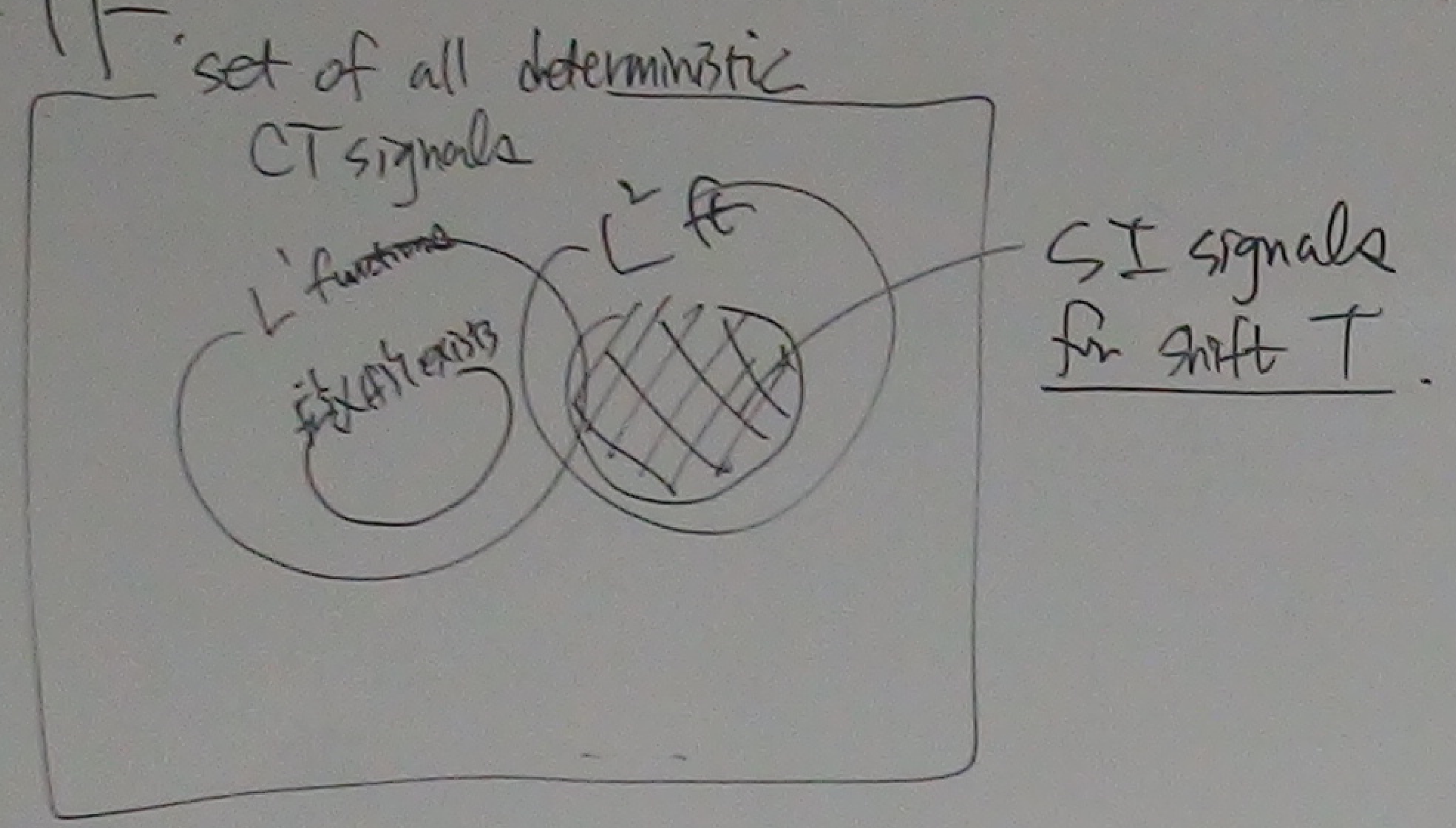
MS periodic ~~cyclostationary~~ $\Rightarrow R_{XX}(t_1, t_2) = R_{XX}(t_1+mT, t_2+nT)$, $\forall t$, $\forall m, n$

?



EECE 695M CM#24b

□ FTF



$SI(T) = \{ x(t) \in L^2 :$

$$\left.\begin{array}{l} x(t) \in SI(T) \\ \Rightarrow x(t+nT) \in SI(T) \end{array}\right\}.$$

$$x(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT)$$

$$(b_n)_n \in \ell^2 \left(\sum_{n=-\infty}^{\infty} |b_n|^2 < \infty \right)$$

□ Generator $p(t) \in L^2$

$$\left\{ x(t) : x(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT),\ (b_n) \in \ell^2 \right\} \subset L^2$$

$$p_1(t),\ p_2(t) \in L^2$$

$$\left\{ x(t) : x(t) = \sum_{k=1}^{2} \sum_{n=-\infty}^{\infty} b_{k,n} p_k(t-nT),\ (b_{k,n}) \in \ell^2 \right\} \subset L^2$$

$$\vdots$$

$$\left(p_k(t)\right)_{k=1}^{K} \in L^2$$

$$\left\{ x(t) : x(t) = \sum_{k=1}^{K} \sum_{n=-\infty}^{\infty} b_{k,n} p_k(t-nT),\ (b_{k,n}) \in \ell^2 \right\} \subset L^2.$$

Q. What properties?

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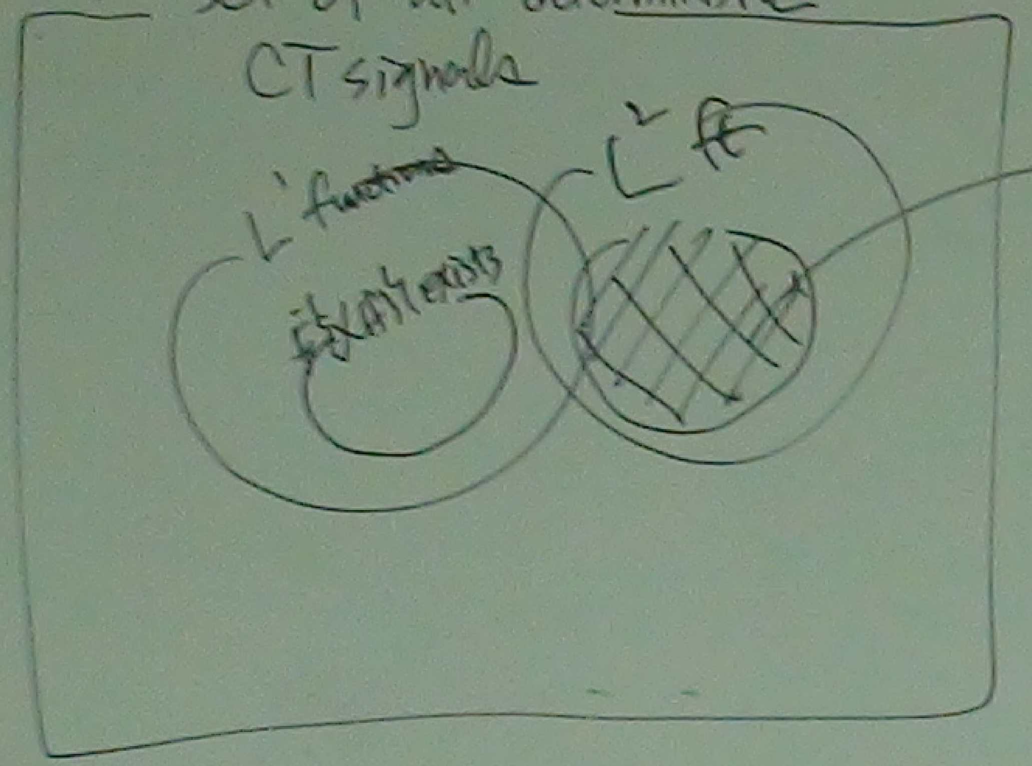
□ FTF

set of all deterministic



SI signals for shift T.

$SI(T) = \{ x(t) \in L^2 :$

$$\left. \begin{array}{l} x(t) \in SI(T) \\ \Rightarrow x(t+nT) \in SI(T) \end{array} \right\}.$$

$$x(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT)$$

$$(b_n)_n \in \ell^2 \left(\sum_{n=-\infty}^{\infty} |b_n|^2 < \infty \right)$$

□ Generator $p(t) \in L^2$

$$\left\{ x(t) : x(t) = \sum_{n=-\infty}^{\infty} b_n p(t-nT),\ (b_n) \in \ell^2 \right\} \subset L^2$$

$$p_1(t),\ p_2(t) \in L^2$$

$$\left\{ x(t) : x(t) = \sum_{k=1}^{2} \sum_{n=-\infty}^{\infty} b_{k,n} p_k(t-nT),\ (b_{k,n}) \in \ell^2 \right\} \subset L^2$$

$$\vdots$$

$$\left(p_k(t) \right)_{k=1}^{K} \in L^2$$

$$\left\{ x(t) : x(t) = \sum_{k=1}^{K} \sum_{n=-\infty}^{\infty} b_{k,n} p_k(t-nT),\ (b_{k,n}) \in \ell^2 \right\} \subset L^2.$$

Q. What properties?

$$\underset{\in SI(B,T)}{x(t)} \rightarrow \boxed{\text{FRESH VEC } (B, 1/T)} \rightarrow x(t) = \sum_{k=1}^{K} \sum_{n=-\infty}^{\infty} b_{k,n} p_k(t-nT)$$

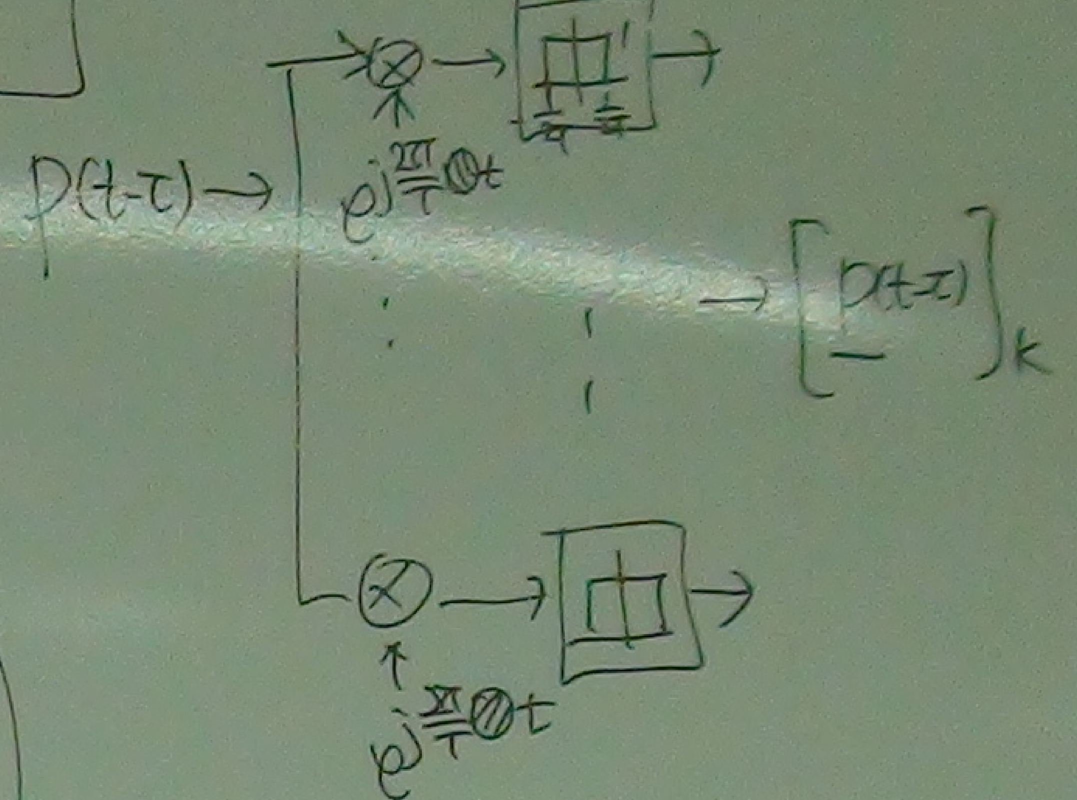
Linear

$$p_k(t) \xrightarrow{F_v} p_k(t)$$

$$p_k(t-nT) \xrightarrow{F_v} p_k(t-nT) \quad (O)$$

$$p_k(t-\tau) \xrightarrow{F_v} p_k(t-\tau) \quad (X)$$

□



$$LHS = \left(p(t-\tau) e^{j\frac{2\pi k}{T}t} \right) * g(t)$$

$$RHS = \left(p(t) e^{j\frac{2\pi k}{T}t} \right) * g(t) \Big|_{t=t-\tau}$$

$$= \left(p(t-\tau) e^{j\frac{2\pi k(t-\tau)}{T}} \right) * g(t)$$

$$LHS = P\left(f - \frac{k}{T}\right) e^{-j2\pi \left(f - \frac{k}{T}\right)\tau} G(f)$$

$$RHS = P\left(f - \frac{k}{T}\right) e^{-j2\pi f \tau} G(f)$$

$$e^{j\frac{2\pi}{T}k\tau} = 1,\ \forall k$$

If $\boxed{\tau = nT}$, then RHS = LHS.

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□ $x(t) \rightarrow \boxed{\begin{matrix} F_v \\ (B, 1/T) \end{matrix}} \rightarrow \underline{x}(t)$

$\in SI(B,T)$

$x(t) = \sum_K \sum_n b_{n,K}\, p_K(t-nT)$

$\underline{x}(t) = \sum_K \sum_n b_{n,K}\, \underline{p}_K(t-nT)$

$\underline{p}_K(f) = \mathcal{F}\{\underline{p}_K(t)\}$ VFT of $\underline{p}_K(t)$

$\mathcal{F}$

$\underline{x}(f) = \sum_K \sum_n b_{n,K}\, \underline{p}_K(f)\, e^{-j2\pi f nT}$

$$\underline{p}_K(t-nT) = \begin{bmatrix} p_{K,1}(t-nT) \\ p_{K,2}(t-nT) \\ \vdots \\ p_{K,2K+1}(t-nT) \end{bmatrix} \leftrightarrow \begin{bmatrix} p_{K,1}(f)e^{-j2\pi f nT} \\ p_{K,2}(f)e^{-j2\pi f nT} \\ \vdots \\ p_{K,2K+1}(f)e^{-j2\pi f nT} \end{bmatrix}$$

□ Nyquist Criterion. < NC   1 generator ; GNC   K   " s.

(i) NC. $x(t) \rightarrow \boxed{p^*(-t)} \rightarrow$ sample @ $t=mT$ $\rightarrow b_m$

$y(t) = x(t) * p^*(-t)$

$\updownarrow$

$Y(f) = X(f)P^*(f)$

$\Leftrightarrow b_m = \int_{-\infty}^{\infty} x(t)\, p^*(t-mT)\, dt$

$= x(t) * p^*(-t)\big|_{t=mT}$

(ii) GNC $x(t)$ — $\boxed{p_1^*(-t)}$ — sample @ $t=mT$ — $b_{m,1}$; $\boxed{p_2^*(-t)}$ — $b_{m,2}$; $\vdots$; $\boxed{p_K^*(-t)}$ — $b_{m,K}$

$\frac{1}{T}\underline{p}_K^H(f)\,\underline{p}_{K'}(f) = \delta[K-K'], \forall f.$

Q. What properties?

$x(t) \rightarrow \boxed{\begin{matrix} \text{FRESH} \\ \text{vec} \\ (B, 1/T) \end{matrix}}$ (Linear) $\rightarrow \underline{x}(t) = \sum_{k=1}^{K} \sum_{n=-\infty}^{\infty} b_{k,n}\, \underline{p}_k(t-nT)$

$\in SI(B,T)$

$p_k(t) \xrightarrow{F_v} \underline{p}_k(t)$

$p_k(t-nT) \xrightarrow{F_v} \underline{p}_k(t-nT)$

$\int_{-\infty}^{\infty} \underline{p}_k(t-mT)\, \underline{p}^*_{k'}(t-nT)\, dt = \delta[k-k']\,\delta[m-m'].$

$\rightarrow P(f) = [\underline{p}_1(f) \cdots \underline{p}_K(f)].$ $\frac{1}{T} P^H(f) P(f) = I_K, \forall f$

$\underline{y}(t) = \mathrm{diag}\{\underline{p}(-t)\}^H * \underline{x}(t)$

$y(t) = \underline{p}^H(-t) * \underline{x}(t)$

$y(mT) = \int_{-\infty}^{\infty} \underline{p}^H(-(t-mT))\, \underline{x}(t)\, dt$

$= \int_{-\infty}^{\infty} \underline{p}^H(f)\, e^{j2\pi f mT}\, \underline{x}(f)\, df$ $\langle \frac{1}{T} \rangle$

$= b_m$

$= \int_{\langle \frac{1}{T} \rangle} \underline{p}^H(f) \boxed{e^{j2\pi f mT}} \sum_{n=-\infty}^{\infty} b_n\, \underline{p}(f) \boxed{e^{-j2\pi f nT}}\, df$

$\Leftrightarrow \sum_{n=-\infty}^{\infty} b_n \underbrace{\int_{\langle \frac{1}{T} \rangle} \left(\underline{p}^H(f)\,\underline{p}(f)\right) e^{-j2\pi f(m-n)T}\, df}_{=\delta[m-n]}$

$\Leftrightarrow \frac{1}{T}\underline{p}^H(f)\,\underline{p}(f) = 1, \forall f$

$\overset{?}{\Leftrightarrow} \frac{1}{T}\sum_K \left|P(f - \frac{K}{T})\right|^2 = 1, \forall f$