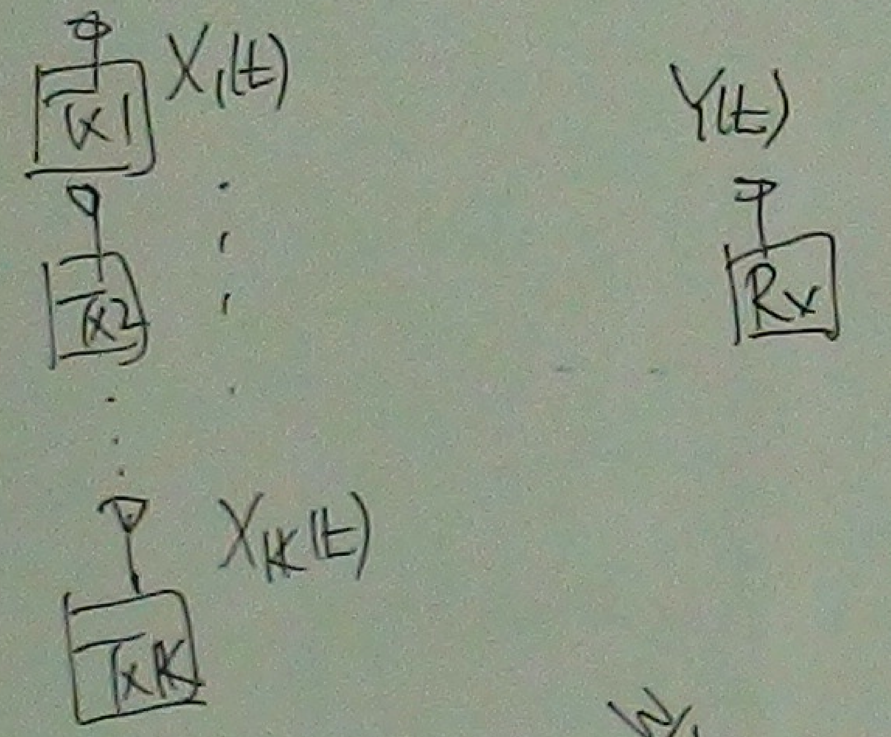
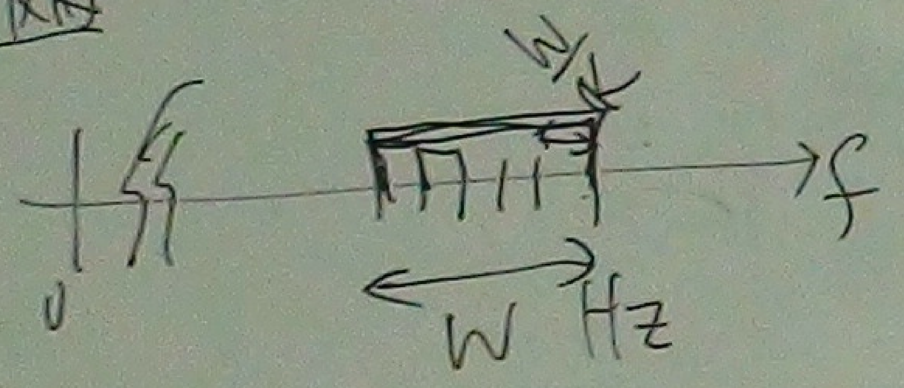
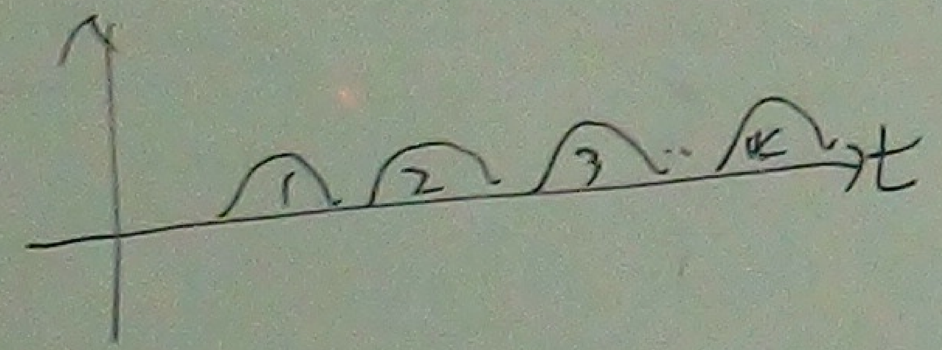
EECE 6935M CM#28a

□ T-IT 2005 "Continuous-Time Equivalents of Welch Bound Equality Sequences"

○ MA (multiple access) < FD, TD, CD $\cong$ DS/SS MA < short-code (←), long-code < generalized random spreading







$$\boxed{Y(t)=\sum_{k=1}^{K}\sum_{m=-\infty}^{\infty}\sqrt{P}\,d_k[m]\,S_k(t-mT)+N(t)}$$

$\underbrace{\|S_k(t)\|^2=T}$, $=\int_{-\infty}^{\infty}|S_k(t)|^2dt=T$, $\int_{-\frac{W}{2}}^{\frac{W}{2}}|S_k(f)|^2df=T$

$(d_k[m])_m$ zero-mean, unit-variance white symbol seq

$\boxed{KT \geq \frac{1}{W}}$

$\{(d_k[m])_m\}_k$ uncorrelated

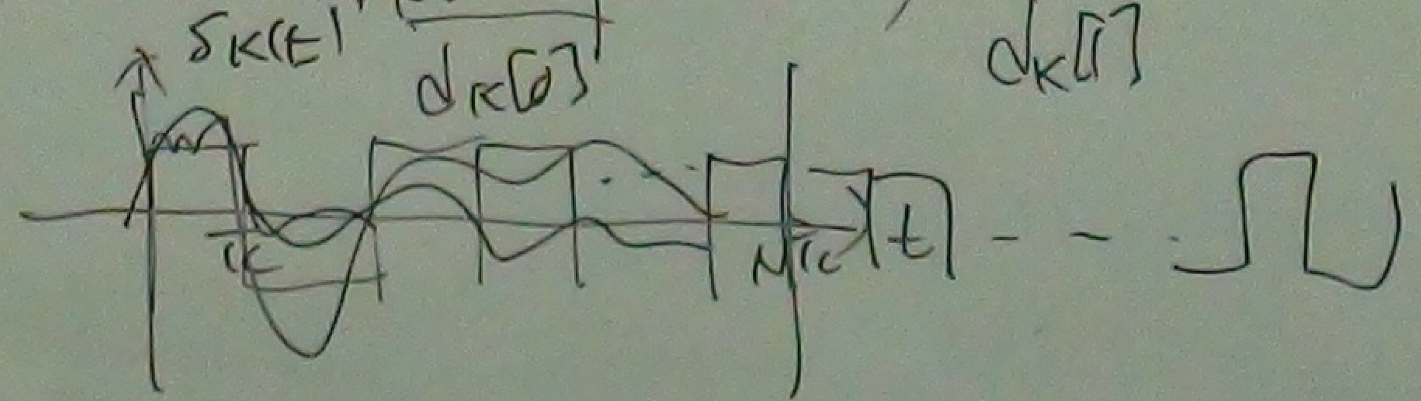
$T=NT_c$

$$P_k=\frac{P\|S_k(t)\|^2}{T}=P,\ \forall k$$

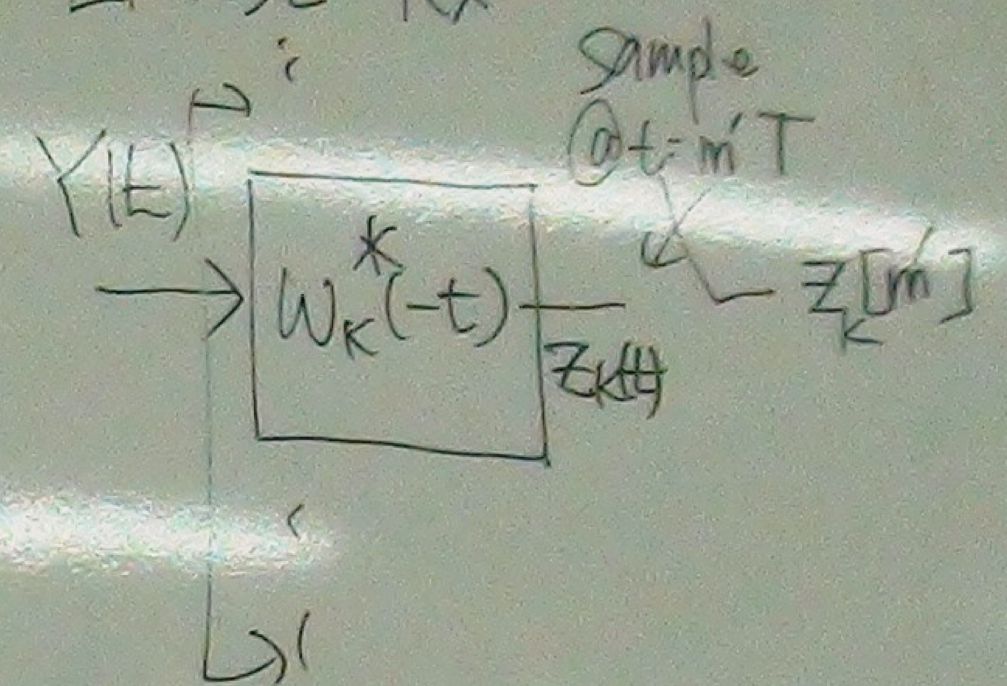
$$S_k(t)=\sum_{n=0}^{N-1} b_k[n]\,p(t-nT_c) \quad \leftarrow \text{given}$$

$=b_k[0]p(t)+b_k[1]p(t-T_c)+\cdots+b_k[N-1]p(t-(N-1)T_c)$

(X)



○ LMMSE Rx



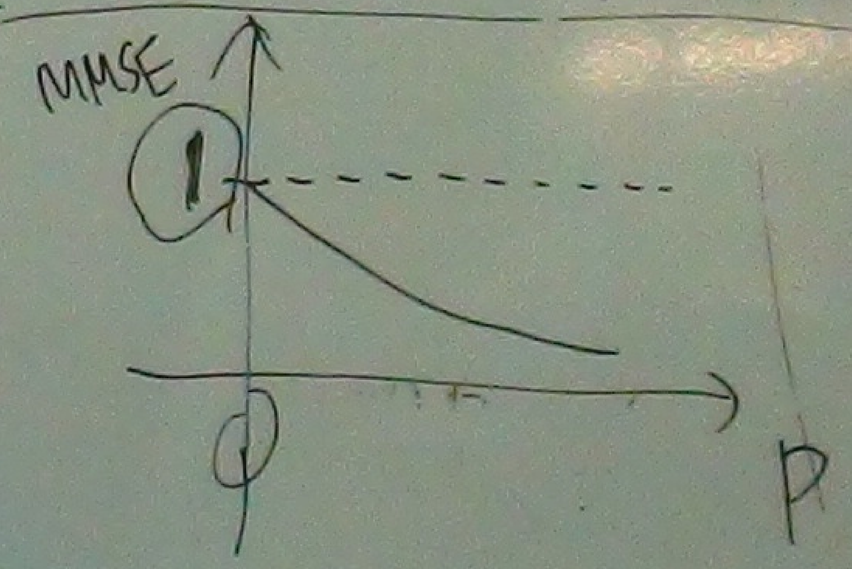
$$\text{TMSE}=\sum_{k=1}^{K}E\{\|Z_k[m']-d_k[m']\|^2\}$$

$$\min_{W_1(t),\ldots,W_K(t)} \text{TMSE}$$

$$\equiv \left\{\min_{W_k(t)} E\{\|Z_k[m']-d_k[m']\|^2\}\right\}_{\forall k}$$

$$\text{MMSE}_k=\frac{1}{\text{MSINR}+1}$$

$$\text{MSINR}_k=\frac{1}{\text{MMSE}_k}-1$$



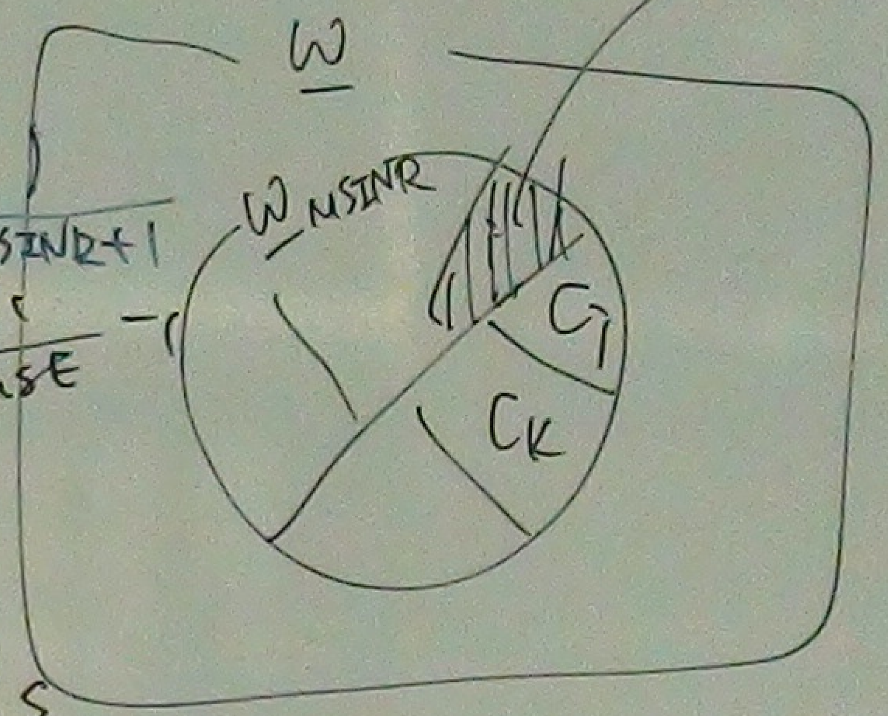
EECE 695M, CM#28 b

☐ T-IT 2005 "Continuous-Time Equivalents of Welch Bound Equality Sequences"

If $\underline{w}_{MSINR}$ maximizes SINR, then $k\,\underline{w}_{MSINR}$ " " $(k \neq 0)$

Not unique!

$|\underline{w}^H \underline{s}|^2 = C_i$



○ MSE vs. SINR

$\underline{Y} = d\,\underline{s} + \underline{N}$

d : random variable

$E\{d\} = 0$

$E\{d^2\} = 0$

$E\{|d|^2\} = 1$

$\underline{s}$: given deterministic

$\underline{N}$: additive noise

$E\{\underline{N}\} = 0$

$E\{\underline{N}\underline{N}^T\} = 0$

$E\{\underline{N}\underline{N}^H\} = R_{NN}$.

$$\min_{\underline{w}} E\{|\underline{w}^H \underline{Y} - d|^2\} = \text{MMSE}$$

$$\underline{w}^H \underline{Y} = \underbrace{d\,\underline{w}^H \underline{s}}_{\text{signal}} + \underbrace{\underline{w}^H \underline{N}}_{\text{IN}}$$

$|d| = 1$. MPSK

$E\{|d|^2\} = 1$

$$\text{SINR} \triangleq \frac{|\underline{w}^H \underline{s}|^2 \leftarrow \text{average signal power}}{E\{|\underline{w}^H \underline{N}|^2\}} = \frac{|\underline{w}^H \underline{s}|^2}{\text{Var}\{\underline{w}^H \underline{Y} \mid d\}}$$

$$\text{MMSE} = \frac{1}{\text{MSINR}+1}$$

$$\text{MSINR} = \frac{1}{\text{MMSE}} - 1$$

$$\boxed{\max_{\underline{w}} \frac{|\underline{w}^H \underline{s}|^2}{\underline{w}^H R_{NN} \underline{w}}}$$

$$\boxed{\min_{\underline{w}} \underline{w}^H (\underline{s}\underline{s}^H + R_{NN})\underline{w} - 2\,\text{Re}\{\underline{w}^H \underline{s}\} + 1}$$

$$\left(1 - \frac{\underline{s}^H R_{NN}^{-1} \underline{s}}{1 + \underline{s}^H R_{NN}^{-1} \underline{s}}\right) R_{NN}^{-1} \underline{s}$$

$$\frac{\partial}{\partial \underline{w}^H} = 0 \Leftrightarrow 2(\underline{s}\underline{s}^H + R_{NN})\underline{w}_{opt} - 2\underline{s} = 0$$

unique!

$$\underline{w}_{opt} = \boxed{(\underline{s}\underline{s}^H + R_{NN})^{-1} \underline{s}} = \left(R_{NN}^{-1} - \frac{R_{NN}^{-1}\underline{s}\,\underline{s}^H R_{NN}^{-1}}{1 + \underline{s}^H R_{NN}^{-1} \underline{s}}\right)\underline{s}$$

$$\text{MMSE} = 1 - \underline{s}^H \boxed{(R_{NN} + \underline{s}\underline{s}^H)^{-1}} \underline{s} = 1 - \underline{s}^H \left(1 - \frac{\cancel{\underline{s}\underline{s}^H R_{NN}^{-1}}}{1 + \underline{s}^H R_{NN}^{-1} \underline{s}}\right) R_{NN}^{-1} \underline{s}$$

$$= 1 - \frac{\underline{s}^H R_{NN}^{-1} \underline{s}}{1 + \underline{s}^H R_{NN}^{-1} \underline{s}}$$

$$= \frac{1}{1 + \underline{s}^H R_{NN}^{-1} \underline{s}} = \frac{1}{\text{MSINR}+1}$$

$$\frac{1}{C+ab} - \frac{1}{C} = \frac{C - (C+ab)}{(C+ab)C} = \frac{-ab}{(C+ab)C} = \frac{-\frac{a}{C}\frac{b}{C}}{\left(1 + \frac{ab}{C}\right)}$$

$$\max_{\underline{w}} \frac{\underline{w}^H (\underline{s}\underline{s}^H) \underline{w}}{\underline{w}^H (R_{NN}) \underline{w}}$$

$$\text{MSINR} = \frac{|\underline{s}^H R_{NN}^{-1} \underline{s}|^2}{\underline{s}^H R_{NN}^{-1} \underline{s}} = \underline{s}^H R_{NN}^{-1} \underline{s}$$

$$= \begin{array}{ll} \min_{\underline{w}} & \underline{w}^H R_{NN} \underline{w} \\ \text{s.t.} & \underline{w}^H \underline{s} = 1 \end{array}$$

$$L = \underline{w}^H R_{NN} \underline{w} - \lambda(\underline{w}^H \underline{s} - 1)$$

$$\left\{\begin{array}{l} \left.\frac{\partial L}{\partial \underline{w}^H}\right|_{\underline{w}_{opt}, \lambda_{opt}} = 0 \Rightarrow 2 R_{NN} \underline{w}_{opt} - \lambda \underline{s} = 0 \\ \left.\frac{\partial L}{\partial \lambda}\right|_{\underline{w}_{opt}, \lambda_{opt}} = 0 \Rightarrow \underline{w}_{opt}^H \underline{s} = 1 \end{array}\right.$$

$$\underline{w}_{opt} = \frac{1}{2}\lambda_{opt} R_{NN}^{-1} \underline{s} \quad // \quad \frac{R_{NN}^{-1}\underline{s}}{\underline{s}^H R_{NN}^{-1} \underline{s}}$$

$$\frac{1}{2}\lambda_{opt}^{*}\, \underbrace{\underline{s}^H R_{NN}^{-1} \underline{s}}_{\text{real}} = 1 \qquad \lambda_{opt} = \frac{2}{\underline{s}^H R_{NN}^{-1} \underline{s}}$$

$$\boxed{\underline{w}_{MSINR} \propto R_{NN}^{-1} \underline{s}}$$

EECE 695M CM#28C

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○ $Y(t) = \sum_{k=1}^{K}\sum_{m=-\infty}^{\infty} d_k[m]\, s_k(t-mT) + N(t)$

$\uparrow$ AWGN

$\equiv \underline{Y}(t) = \sum_{k=1}^{K}\sum_{m=-\infty}^{\infty} d_k[m]\, \underline{s}_k(t-mT) + \underline{N}(t)$

$$\underline{w}_k(t) \longleftrightarrow \underline{w}_k(f) = \underline{s}_k(f)^H \underbrace{\left(\sigma^2 I + \sum_{k=1}^{K} \underline{s}_k(f)\underline{s}_k(f)^H\right)}_{R_{YY}(f)}^{-1} \underline{s}_k(f)$$

$$MMSE_k = 1 - \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \underline{s}_k(f)^H (\quad)^{-1} \underline{s}_k(f)\, df$$

$$TMSE = \sum_{k=1}^{K} MMSE_k = K - \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \underline{\left[\sum \underline{s}_k(f)^H (\quad)^{-1} \underline{s}_k(f)\right]} df.$$

$$(\sigma^2 I + \Lambda)^{-1}\Lambda = \frac{\lambda_k}{\sigma^2 + \lambda_k}$$

○ $\min_{s_1(t), \dots, s_K(t)} TMSE = K - T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{l=1}^{L} \frac{\lambda_l(f)}{\lambda_l(f) + \sigma^2} df$

$\Rightarrow \boxed{S(f)S(f)^H = \left(\frac{K}{N}\right) T I}$ $\forall f$, ~~$K \geq N$~~

s.t. $\|s_k(t)\|^2 = T,\ \forall k$

$$\int_{-\frac{W}{2}}^{\frac{W}{2}} |\underline{s}_k(f)|^2 df = T.$$

$tr(ABC) = tr(BCA)$

$\boxed{K < N:\ S(f)^H S(f) = I}$

○ $\sum_{k=1}^{K} \boxed{\underline{s}_k^H \left(\sigma^2 I + \sum \underline{s}_k \underline{s}_k^H\right)^{-1} \underline{s}_k}$ scalar

$= \sum_{k=1}^{K} tr\left(\left(\sigma^2 I + \sum \underline{s}_k \underline{s}_k^H\right)^{-1} \underline{s}_k \underline{s}_k^H\right)$

$= tr\left\{\left(\sigma^2 I + \boxed{\sum_{k=1}^{K} \underline{s}_k \underline{s}_k^H}\right)^{-1} \boxed{\left(\sum_{k=1}^{K} \underline{s}_k \underline{s}_k^H\right)}\right\}$ $L \times K$

where $\sum_{k=1}^{K} \underline{s}_k \underline{s}_k^H = \underbrace{[\underline{s}_1\, \underline{s}_2 \cdots \underline{s}_K]}_{\triangleq S} \begin{bmatrix} \underline{s}_1^H \\ \underline{s}_2^H \\ \vdots \\ \underline{s}_K^H \end{bmatrix} = SS^H.$

$= U\Lambda U^H$

$= tr\left\{\left(\sigma^2 I + U\Lambda U^H\right)^{-1} U\Lambda U^H\right\} = \sum_{l=1}^{L} \frac{\lambda_l}{\sigma^2 + \lambda_l}$